MATH 207(c)–Analysis IV

1. **(a)** Consider the function defined by

$$f(x) = e^{-ix}\big(\cos(x) + i\sin(x)\big)$$

for $x \in \mathbb{R}$. Compute the derivative of f (the function $x \to e^{\lambda x}$ satisfies the same differentiation properties when $\lambda \in \mathbb{C}$ as in the case $\lambda \in \mathbb{R}$). Show that f is constant and equal to 1 and, thus, verify Euler's formula.

(b) Show that, for any $\alpha, \beta \in \mathbb{R}$:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

(Hint: you might want to use Euler's formula for $e^{i(\alpha+\beta)}$.)

2. Calculate the real and imaginary parts of the following expressions:

(a) $(1+2i)(2-3i)$ **(c)** $(1+i)^3 + (1-i)^3$ **(e)** $\left(\frac{1}{i}\right)^{2025}$ **(g)** $\frac{2i^{19}-10i^{12}}{1+i}$

(b) $\frac{1-3i}{1+i}$ **(d)** $\frac{1}{1+i} + \frac{2}{1-i}$ **(f)** $e^{-1025\pi i}$ **(h)** $\big(1+\sqrt{3}i\big)^{10}$

3. Calculate the modulus and an argument of the following expressions:

(a) $5+5i$ **(c)** $\frac{1+\sqrt{3}i}{1-i}$

(b) $(-1+\sqrt{3}i)^{10}$ **(d)** 3^i

4. Determine all the complex solutions of the following equations:

(a) $z^5 = 1$ **(c)** $z^2 - z + 2 = 0$ **(e)** $\frac{1}{z-i} + \frac{1}{z^2-1} = 0$

(b) $z^4 = 4+4i$ **(d)** $z^4 - 2z^2 + i = 0$ **(f)** $|z-1| = |z+1|$

5. Show that, for any $x \in \mathbb{R}$, the complex number

$$z = \frac{x+i}{x-i}$$

lies on the unit circle. Show also that every point on the unit circle except for $z=1$ can be expressed in the above form.

6. Characterize geometrically the following subsets of $\mathbb{C}$:

(a) $\Big\{z \in \mathbb{C} : \quad |z-1| = 1\Big\}$

(b) $\Big\{z \in \mathbb{C} : \quad \frac{|z-1|}{|z-i|} = 1\Big\}$

(c) $\Big\{z \in \mathbb{C} : \quad |z-1| + |z+1| = 3\Big\}$

(d) $\Big\{z \in \mathbb{C} : \quad z = 1 + 2t + 4t^2 i \quad \text{for} \;\; t \in \mathbb{R}\Big\}$

7. Show the following equality between sets:

$$\Big\{z \in \mathbb{C}^* : \quad z + \frac{1}{z} \in \mathbb{R}\Big\} = \Big\{z \in \mathbb{C}^* : \quad \mathrm{Im}(z) = 0 \quad \text{or} \quad |z| = 1\Big\}.$$